How delay equations arise in Engineering?

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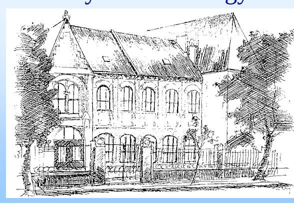


Contents

Answer: Delay equations arise in Engineering…
… by the contact of bodies, and
by the information system of control.

- Linear stability and bifurcations – summary
- Machine tool vibrations
- Shimmying wheels of trucks and motorcycles
- Balancing – human and robotic
- Robotic position and force control

Stability of linear RFDEs of n DoF systems

Delayed mechanical systems include 2nd derivatives:

$$M\ddot{x}(t)+\int_{-h}^{0}\mathrm{d}_{\vartheta}B(t,\vartheta)\dot{x}(t+\vartheta)+\int_{-h}^{0}\mathrm{d}_{\vartheta}K(t,\vartheta)x(t+\vartheta)=0$$

Autonomous systems: $B(t,\vartheta)\equiv B(\vartheta),\ K(t,\vartheta)\equiv K(\vartheta)$

Trial solution: $\boxed{x(t)=A\mathrm{e}^{\lambda t}}$ $\quad A\in R^n$

Characteristic roots: Re $\lambda_j<0$, j=1,2,… ⇔ stability

$$D(\lambda)=\det\left(M\lambda^2+\int_{-h}^{0}\lambda\mathrm{e}^{\lambda\vartheta}\mathrm{d}_{\vartheta}B(t,\vartheta)+\int_{-h}^{0}\mathrm{e}^{\lambda\vartheta}\mathrm{d}K(\vartheta)\right)$$

D-curves: $R(\omega)=\mathrm{Re}\,D(i\omega),\ S(\omega)=\mathrm{Im}\,D(i\omega),\ \omega\in[0,\infty)$

$$\left.\begin{array}{l} R(\rho_k)=0,k=1,\dots,r:\ \ S(\rho_k)\neq 0,k=1,\dots,r \\ \sum_{k=1}^{r}(-1)^k\,\mathrm{sgn}\,S(\rho_k)=(-1)^n n \end{array}\right\}\Leftrightarrow \text{stability}$$

Examples with 1 DoF, n = 1

$$\ddot{x}(t)+c_0x(t)=c_1\int_{-1}^{0}w(\vartheta)x(t+\vartheta)\mathrm{d}\vartheta,\quad w(\vartheta)\equiv 1$$

$$D(\lambda)=\lambda^2+c_0-c_1\int_{-1}^{0}e^{\lambda\vartheta}\mathrm{d}\vartheta=\lambda^2+c_0-c_1\frac{1-e^{-\lambda}}{\lambda}$$

$$R(\omega)=-\omega^2+c_0-c_1\frac{\sin\omega}{\omega}\quad\Rightarrow\lim_{\omega\to+\infty}R(\omega)=-\infty$$

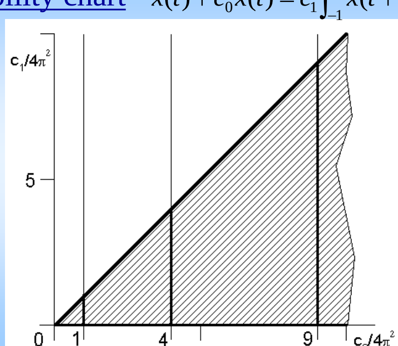
$$S(\omega)=c_1\frac{1-\cos\omega}{\omega}\quad\Rightarrow S(\omega)>0\ \text{for}\ \boxed{c_1>0},$$

$$\omega\neq 2k\pi,k=0,1,\dots$$

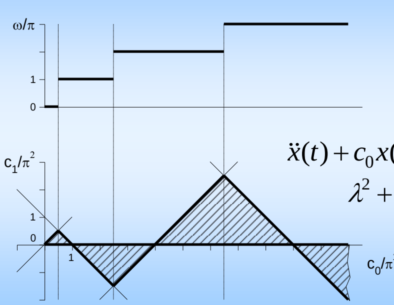
$$S(\rho_k)\neq 0,k=1,\dots,r\ \Rightarrow R(2k\pi)=\boxed{-4k^2\pi^2+c_0\neq 0}$$

$$\sum_{k=1}^{r}(-1)^k\underbrace{\mathrm{sgn}\,S(\rho_k)}_{+1}=\underbrace{(-1)^n n}_{-1}\quad\Rightarrow r\ \text{odd}\quad\Rightarrow \mathrm{R}(0)=\boxed{c_0-c_1>0}$$

Stability chart $\ddot{x}(t)+c_0x(t)=c_1\int_{-1}^{0}x(t+\vartheta)\mathrm{d}\vartheta$



Delayed oscillators $w(\vartheta)=\delta(\vartheta+1)$



vibration frequencies

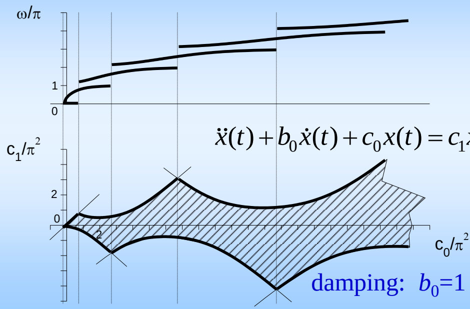
$$\ddot{x}(t)+c_0x(t)=c_1x(t-1)$$

$$\lambda^2+c_0-c_1\mathrm{e}^{-\lambda}=0$$

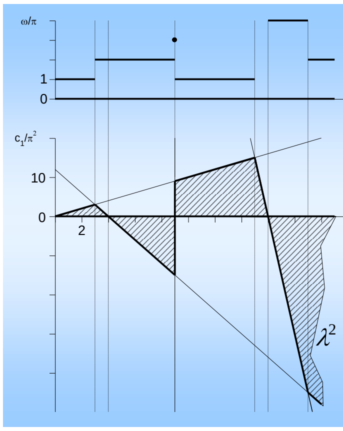
stability chart

Delayed oscillator with damping

$$\ddot{x}(t)+b_0\dot{x}(t)+c_0x(t)=c_1x(t-1)$$



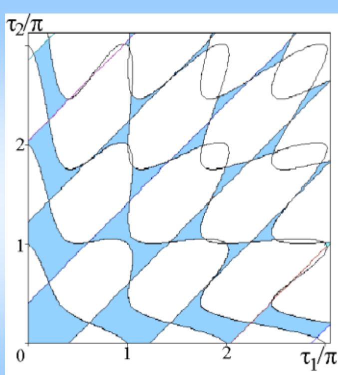
damping: b_0=1



$$\ddot{x}(t)+c_0x(t)= c_1\int_{-1}^{0} w(\vartheta)x(t+\vartheta)\,\mathrm{d}\vartheta$$

$$w(\vartheta)=-\frac{\pi}{2}\sin(\pi\vartheta)$$

$$\lambda^2+c_0-c_1\frac{\pi^2}{2}\frac{1+\mathrm{e}^{-\lambda}}{\lambda^2+\pi^2}=0$$



$$\ddot{x}(t)+6x(t)= x(t-\tau_1)+x(t-\tau_2)$$

Non-autonomous linear RFDEs

$$M\ddot{x}(t)+\int_{-h}^{0}\mathrm{d}_\vartheta B(t,\vartheta)\dot{x}(t+\vartheta)+\int_{-h}^{0}\mathrm{d}_\vartheta K(t,\vartheta)x(t+\vartheta)=0$$

Time-periodic systems: $B(t+T,\vartheta)=B(t,\vartheta)$

Trial solution: $\boxed{x(t)=p(t)\mathrm{e}^{\lambda t}}$ $K(t+T,\vartheta)=K(t,\vartheta)$

$$p(t+T)=p(t)=\sum_{k=0}^{+\infty}(A_k\cos(k\tfrac{2\pi}{T}t)+B_k\sin(k\tfrac{2\pi}{T}t))$$

Hill's infinite dimensional determinant $\Rightarrow$

characteristic function $\Rightarrow$ characteristic roots λ

Re $\lambda_j<0$, j=1,2,… $\Leftrightarrow$ stability $\Leftrightarrow$ $|\mu_j|<1$, j=1,2,…

for characteristic multipliers $\mu=\mathrm{e}^{\lambda T}$ of fund. op. at T

The delayed Mathieu equation

$$\ddot{x}(t)+(\delta+\varepsilon\cos t)x(t)=b\,x(t-2\pi)$$

$$x(t)=\sum_{k=0}^{\infty}\left(A_k\,\mathrm{e}^{\mathrm{i}kt}+B_k\,\mathrm{e}^{-\mathrm{i}kt}\right)\mathrm{e}^{\lambda t}+\sum_{k=0}^{\infty}\left(\bar{A}_k\,\mathrm{e}^{-\mathrm{i}kt}+\bar{B}_k\,\mathrm{e}^{\mathrm{i}kt}\right)\mathrm{e}^{\bar{\lambda}t}$$

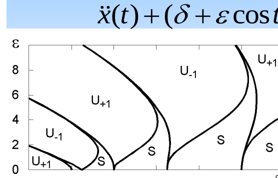
$$x(t)=\sum_{k=-\infty}^{\infty}C_k\,\mathrm{e}^{(\lambda+\mathrm{i}k)t}+\bar{C}_k\,\mathrm{e}^{(\bar{\lambda}-\mathrm{i}k)t}$$ Harmonic balance

$\Rightarrow$ Hill's determinant

$$\det\begin{pmatrix}\ddots & & \ddots & & \ddots & 0 & 0\ldots\\ \ldots 0 & \frac{\varepsilon}{2} & & \delta+(\lambda+\mathrm{i}k)^2-b\,\mathrm{e}^{-2\pi(\lambda+\mathrm{i}k)} & & \frac{\varepsilon}{2} & 0\ldots\\ \ldots 0 & 0 & \ddots & & \ddots & & \ddots\end{pmatrix}=0$$

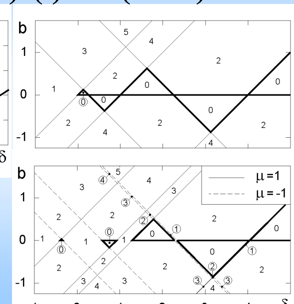
The delayed Mathieu – stability charts

$$\ddot{x}(t)+(\delta+\varepsilon\cos t)x(t)=b\,x(t-2\pi)$$



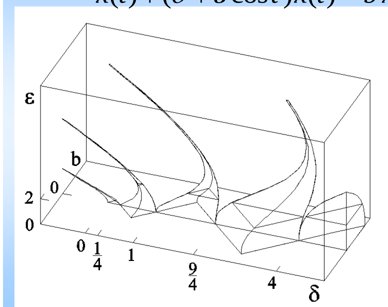
b=0 (Strutt-Incze, 1928)

ε=1



Stability chart of delayed Mathieu

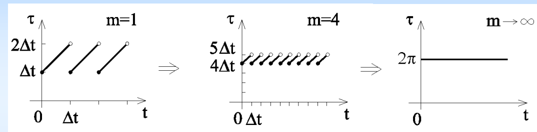
$$\ddot{x}(t)+(\delta+\varepsilon\cos t)x(t)=b\,x(t-2\pi)$$



Insperger, Stépán (2002)

Semi-discretization method – introduction

$$\ddot{x}(t)+c_0x(t)=c_1x(t-\tau)\qquad \tau=2\pi$$



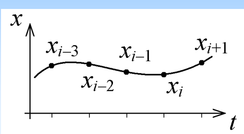
The approximating DDE is *non-autonomous*

$$\ddot{x}(t)+c_0x(t)=c_1x(t-\tau(t)),\quad \tau(t)=t+(m-\mathrm{int}(t/\Delta t))\Delta t$$

$$t\in[t_i,t_{i+1})=[i\Delta t,(i+1)\Delta t)\qquad \Delta t=2\pi/(m+1/2)$$

$$\Rightarrow\ x(t-\tau(t))\equiv x((i-m)\Delta t)=x_{i-m}$$

Introduction to SDM – delayed oscillator



$$\ddot{x}(t)+c_0x(t)=c_1x_{i-m}$$
$$x(t_i)=x_i$$
$$\dot{x}(t_i)=\dot{x}_i$$

$$x(t)=K_{1i}\cos(\sqrt{c_0}t)+K_{2i}\sin(\sqrt{c_0}t)+c_1x_{i-m}/c_0$$

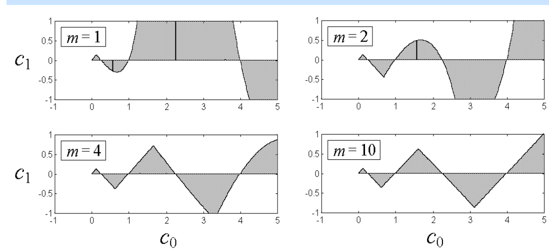
$$\dot{x}(t)=-K_{1i}\sqrt{c_0}\sin(\sqrt{c_0}t)+K_{2i}\sqrt{c_0}\cos(\sqrt{c_0}t)$$

$$x_{i+1}=a_{00}\,x_i+a_{01}\,\dot{x}_i+a_{0m}\,x_{i-m}\qquad \mathbf{y}_i=\mathrm{col}(\dot{x}_i\ x_i\ x_{i-1}\dots x_{i-m})$$
$$\dot{x}_{i+1}=a_{10}\,x_i+a_{11}\,\dot{x}_i+a_{1m}\,x_{i-m}\qquad \boxed{\mathbf{y}_{i+1}=\mathbf{A}\mathbf{y}_i}$$

$$\det(\mu\mathbf{I}-\mathbf{A})=0\ \Rightarrow\ |\mu_{1,2,\dots,m+2}|<1\Leftrightarrow$$ stability

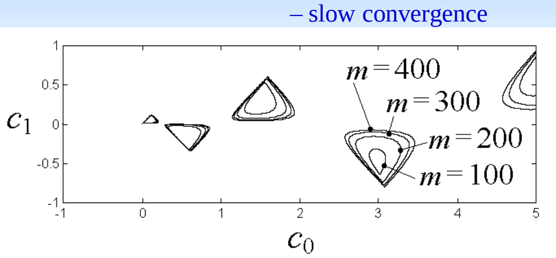
Delayed oscillator – stability chart by SDM

$$\ddot{x}(t)+c_0x(t)=c_1x(t-\tau(t)),\quad \tau(t)=t+(m-\mathrm{int}(t/\Delta t))\Delta t$$



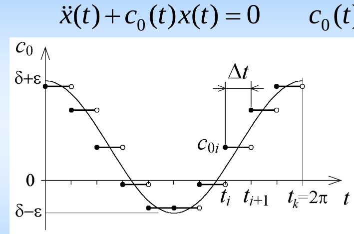
Full discretization - comparison

Discretization also w.r.t. time derivatives
– slow convergence



Introduction to SDM – Mathieu equation

$$\ddot{x}(t)+c_0(t)x(t)=0\qquad c_0(t)=\delta+\varepsilon\cos t$$

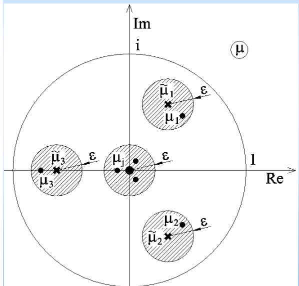


$$t\in[t_i,t_{i+1})$$
$$\ddot{x}(t)+c_{0i}x(t)=0$$
$$x(t_i)=x_i$$
$$\dot{x}(t_i)=\dot{x}_i$$
$$i=0,1,\dots,k-1$$

$$x(t)=x_i\cos\left(\sqrt{c_{0i}}(t-t_i)\right)+\frac{\dot{x}_i}{\sqrt{c_{0i}}}\sin\left(\sqrt{c_{0i}}(t-t_i)\right)$$

Semi-discretization – general case

$$\dot{x}(t)=\int_{-\tau}^{0}\mathrm{d}_{\vartheta}\eta(t,\vartheta)x(t+\vartheta),\quad \eta(t+T,\vartheta)=\eta(t,\vartheta)$$



$\forall\varepsilon>0,\ \exists M(\varepsilon),\ \forall m>M(\varepsilon)$

$\Rightarrow$

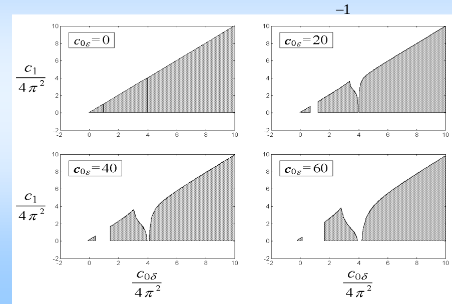
$\mu_j\in\bigcup_{j=1}^{mn}S_{\tilde{\mu}_j,\varepsilon},\ j=1,\dots,mn$

$|\mu_j|<\varepsilon,\qquad j=mn+1,\dots$

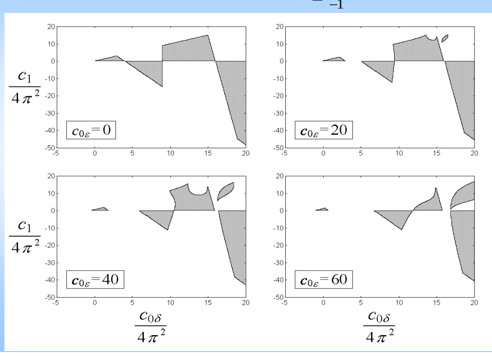
Insperger, Stepan: *Int. J. of Numerical Methods in Engineering* (2002)

Examples – test on delayed Mathieu

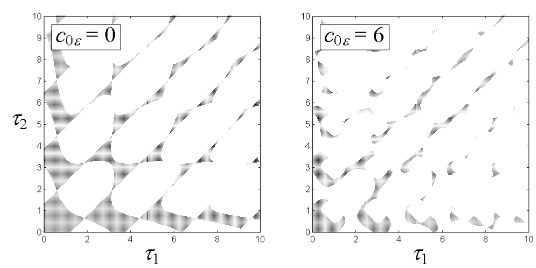
$$\ddot{x}(t)+(c_{0\delta}+c_{0\varepsilon}\cos(4\pi t))x(t)=c_1\int_{-1}^{0}x(t+\vartheta)\mathrm{d}\vartheta$$



$$\ddot{x}(t)+(c_{0\delta}+c_{0\varepsilon}\cos(4\pi t))x(t)=c_1\frac{\pi}{2}\int_{-1}^{0}\sin(\pi\vartheta)x(t+\vartheta)\mathrm{d}\vartheta$$



$$\ddot{x}(t)+(6+c_{0\varepsilon}\cos(2\pi t))x(t)=x(t-\tau_1)+x(t-\tau_2)$$



Nonlinear RFDEs in Engineering

Stability analysis of steady-states is followed by bifurcation analysis

Hopf bifurcation – self-excited vibrations

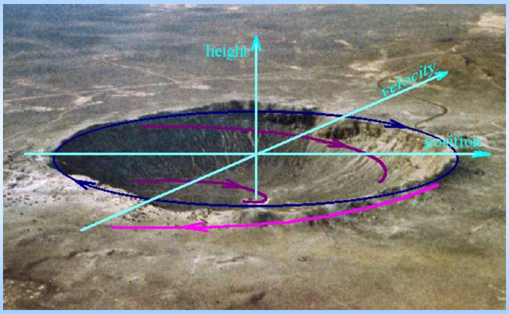
Supercritical case: easy to avoid vibrations by knowing the linear stability behaviour

Subcritical case: the unstable periodic solutions mean a limited domain of attraction for the desired steady-state behaviour – *cannot* be predicted by linear stability analysis.

Stick&slip – unstable periodic motion

akad_csu.mpg

Unstable limit cycle – “ghost” vibration



1. Chatter

~ (high frequency) machine tool vibration

“… Chatter is the most obscure and delicate of all problems facing the machinist – probably ***no rules or formulae*** can be devised which will accurately guide the machinist in taking maximum cuts and speeds possible without producing chatter.”

(Taylor, 1907).

Efficiency of cutting

Specific amount of material cut within a certain time

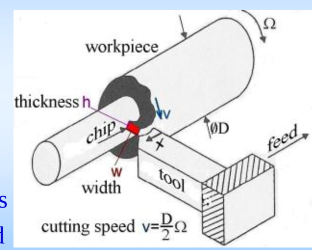
$$\dot{V} = wh\Omega\frac{D}{2}$$

where

w – chip width

h – chip thickness

Ω ~ cutting speed



Efficiency of cutting

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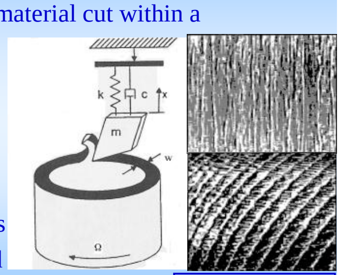


surface quality

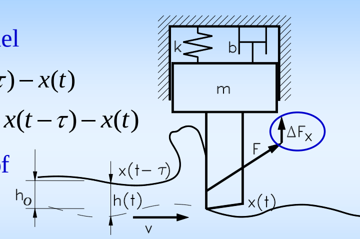
Modelling – regenerative effect

Mechanical model

$h(t) = h_0 + x(t-\tau) - x(t)$

$\Delta h = h(t) - h_0 = x(t-\tau) - x(t)$

τ – time period of revolution

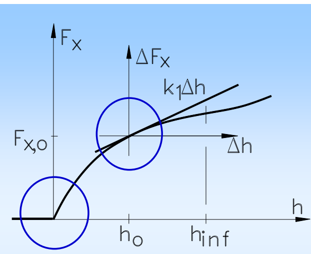


Mathematical model

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = \frac{1}{m}\Delta F_x(\Delta h)$$

Cutting force

¾ rule for nonlinear cutting force



$$F_x(w,h) = c_1 w h^{3/4}$$

$$F_x = F_{x,0} + k_1\Delta h + k_2(\Delta h)^2 + k_3(\Delta h)^3 + \ldots$$

Cutting coefficient

$$k_1(w,h_0) = \left.\frac{\partial F_x(w,h)}{\partial h}\right|_{h_0} = \frac{3}{4} c_1 w h_0^{-1/4}$$

$$k_2 = -\frac{1}{8}\frac{k_1}{h_0} \qquad k_3 = \frac{5}{96}\frac{k_1}{h_0^2}$$

Linear analysis – stability

$$\ddot x(t) + 2\xi\omega_n \dot x(t) + \left(\omega_n^2 + \frac{k_1}{m}\right)x(t) = \frac{k_1}{m}x(t-\tau)$$

Dimensionless time $\tilde t = \omega_n t$

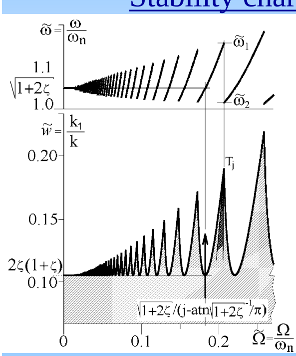
$$x''(\tilde t) + 2\xi x'(\tilde t) + (1+\tilde w)x(\tilde t) = \tilde w x(\tilde t - \omega_n\tau)$$

Dimensionless chip width $\tilde w = \frac{k_1}{m\omega_n^2} = \frac{k_1}{k}$

Dimensionless cutting speed

$$\tilde\Omega = \frac{2\pi}{\tilde\tau} = \frac{2\pi}{\omega_n\tau} = \frac{2\pi}{\omega_n \frac{2\pi}{\Omega}} = \frac{\Omega}{\omega_n}$$

Stability chart of turning



Better stability properties experienced at low and high cutting speeds!

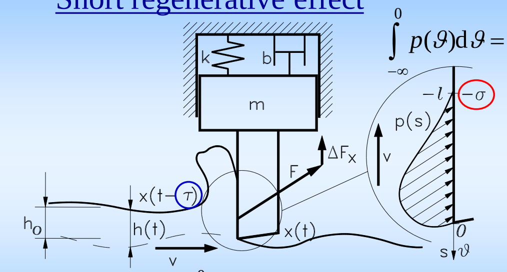
$$\frac{\Omega_j}{\omega_n} = \frac{\sqrt{1+2\xi}}{j - \frac{1}{\pi}\,\mathrm{atn}\,\frac{1}{\sqrt{1+2\xi}}}$$

$$\tilde w_{cr} = 2\xi(1+\xi)$$

$$\omega_{cr} = \omega_n\sqrt{1+2\xi}$$

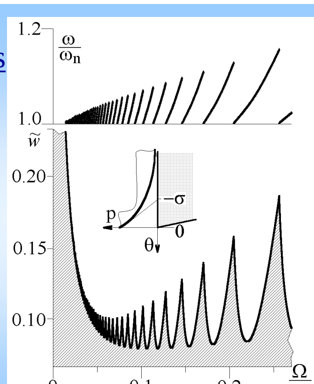
Short regenerative effect

$$\int_{-\infty}^{0} p(\vartheta)\,\mathrm{d}\vartheta = 1$$



$$\ddot x + 2\xi\omega_n\dot x + \omega_n^2 x = \frac{k_1}{m}\int_{-\sigma}^{0} p(\vartheta)\big(x(t+\vartheta-\tau) - x(t+\vartheta)\big)\,\mathrm{d}\vartheta$$

Weight functions



$$p(\vartheta) = \frac{1}{\sigma}e^{\vartheta/\sigma},$$

$$\sigma = q\tau$$

$$q = \frac{\sigma}{\tau} = \frac{l}{2\pi D} = 0.01$$

$$\xi = 0.05$$

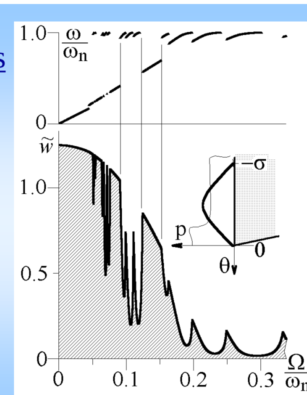
Weight functions

$$p(\vartheta) = -\frac{\pi}{2\sigma}\sin\left(\frac{\pi}{\sigma}\vartheta\right)$$

$$q = \frac{\sigma}{\tau} = \frac{l}{2\pi D} = 0.2$$

$$\xi = 0.01$$

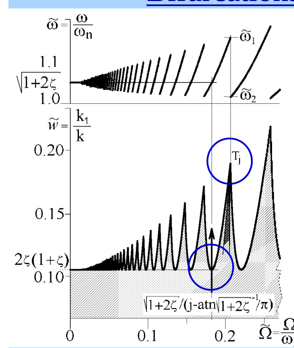
Drilling – low speed, vibrations at very low frequencies!



Drilling

Boeing (2001)

Bifurcations of turning



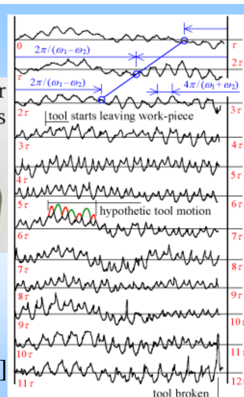
Subcritical Hopf bifurcation: *unstable vibrations* around stable cutting

$$\frac{\Omega_j}{\omega_n} = \frac{\sqrt{1+2\xi}}{j - \frac{1}{\pi}\,\text{atn}\,\frac{1}{\sqrt{1+2\xi}}}$$

$$\tilde{w}_{cr} = 2\xi(1+\xi)$$

$$\omega_{cr} = \omega_n\sqrt{1+2\xi}$$

Machined surface



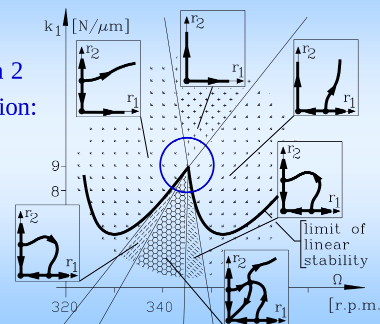
D=176 [mm], τ=0.175 [s]

$$\frac{f_1 + f_2}{2} = \frac{15.3}{\tau} = 88.0[\text{Hz}]$$

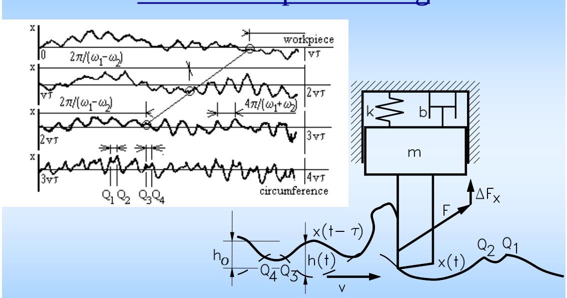
$$\frac{f_1 - f_2}{2} = \frac{15.3}{(2\times 12.5)\tau} = 3.5[\text{Hz}]$$

Unstable quasi-periodic vibration

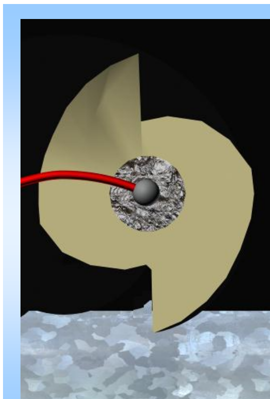
Co-dimension 2 Hopf bifurcation:



Self-interrupted cutting



High-speed milling



Parametrically interrupted cutting

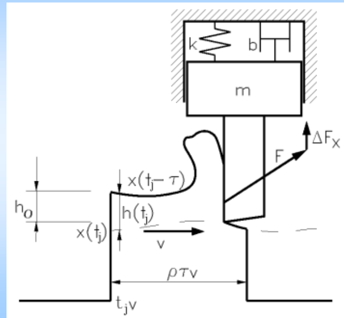
Low number of edges

Low immersion

Highly interrupted

Modelling high-speed milling

Two dynamics:
- free-flight
- cutting with regenerative effect



Nonlinear discrete map of HS milling

$$\begin{bmatrix} x_j \\ v_j \end{bmatrix} = \mathbf{A}\begin{bmatrix} x_{j-1} \\ v_{j-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \sum_{h+k=2,3;\; h,k\ge 0} b_{hk}\, x_{j-1}^h v_{j-1}^k \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\rho\tau}{m} F_0 \end{bmatrix}$$

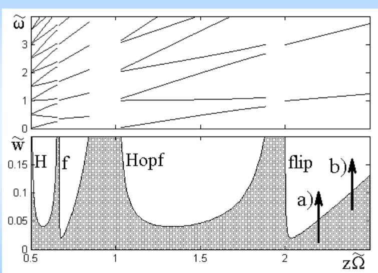
Linear stability: critical characteristic multipliers

$$\mu_1 = -1, \quad \mu_2 = e^{-\zeta\omega_n\tau}\left(\mathrm{sh}(\zeta\omega_n\tau) + \cos(\omega_d\tau)\right)$$

at

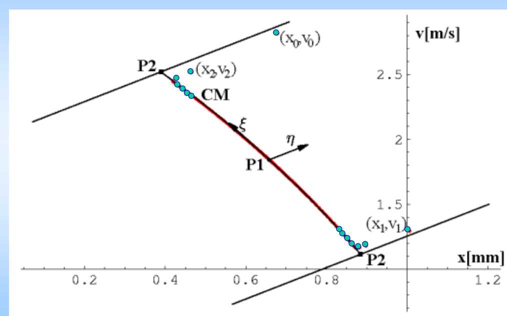
$$\tilde{w}\big|_{cr} = \frac{\rho\tau}{m\omega_d} k_1\big|_{cr} = \frac{\mathrm{ch}(\zeta\omega_n\tau) + \cos(\omega_d\tau)}{\sin(\omega_d\tau)}$$

Stability chart of H-S milling

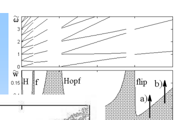


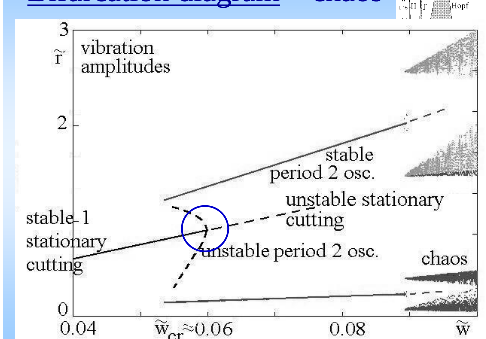
Sense of the period doubling (or flip) bifurcation?

Subcritical flip bifurcation

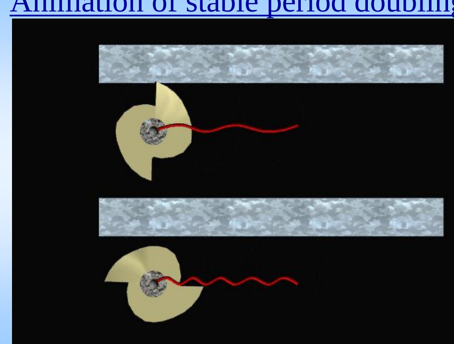


Bifurcation diagram – chaos

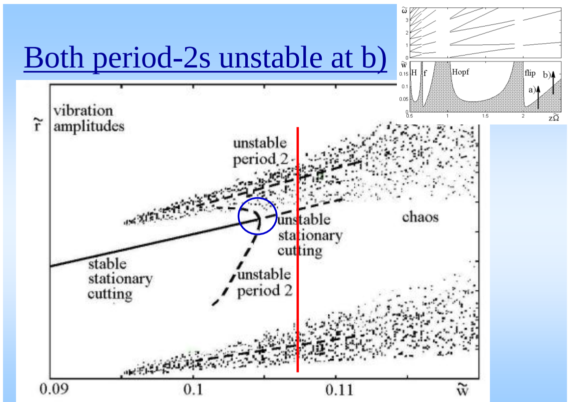




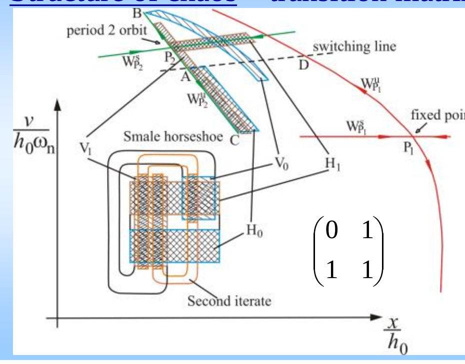
Animation of stable period doubling



Both period-2s unstable at b)



Structure of chaos – transition matrix

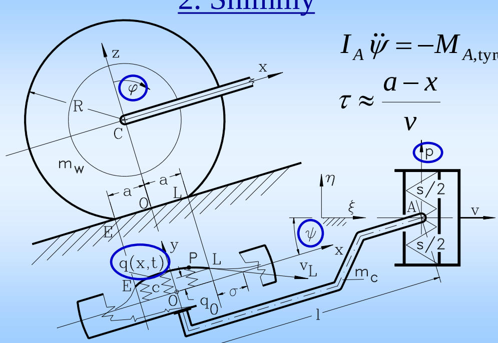


$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

2. Shimmy

$$I_A \ddot{\psi} = -M_{A,\text{tyre}}$$

$$\tau \approx \frac{a-x}{v}$$



Governing equations & memory effect

$$I_A\ddot{\psi}(t) = -c\int_{-a}^{a}(l-x)q(x,t)\,\mathrm{d}x$$

$$\dot{q}(x,t) = v\psi(t) + (l-x)\dot{\psi}(t) + q'(x,t)v + \text{h.o.t.}$$

$$x\in[-a,a],\quad t\in[t_0,\infty),\quad \text{and}\quad q(a,t)=0$$

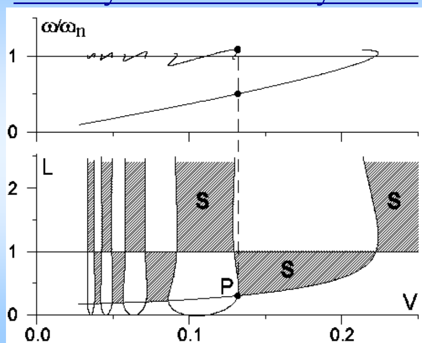
Travelling wave solution of the PDE:

$$q(x,t) = (a-x)\psi(t) + (l-a)\left(\psi(t)-\psi\left(t-\underbrace{\tfrac{a-x}{v}}_{=\tau}\right)\right)+\ldots$$

$$V^2\ddot{\psi}(t)+\psi(t) = \frac{L-1}{L^2+1/3}\int_{-1}^{0}(L-1-2\vartheta)\psi(t+\vartheta)\,\mathrm{d}\vartheta+\ldots$$

$$V=\frac{v}{2a\omega_n},\quad L=\frac{l}{a},\quad \omega_n=\frac{2ac(l^2+a^2/3)}{I_A}$$

Stability chart of shimmy model



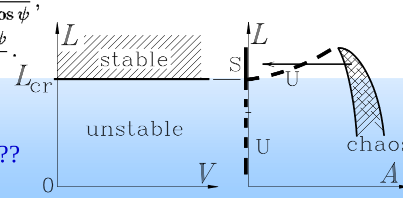
Nonlinear vibrations – without delay

$$\dot\psi = \Omega\,,$$

$$\dot\Omega = -\frac{\frac{v}{l}\left(\frac{1}{\cos^2\psi} - \frac{1}{2} + \frac{3m_w}{2m_c}\tan^2\psi\right)\Omega + \frac{s}{lm_c}p + \left(1+\frac{3m_w}{2m_c}\right)\frac{\tan\psi}{\cos\psi}\Omega^2}{\left(\frac{1}{3}+\tan^2\psi\right)\cos\psi + \frac{m_w}{4m_c}\left(\frac{R^2}{l^2}\cos\psi + 6\tan^2\psi\cos\psi\right)}$$

$$\dot p = v\tan\psi + \frac{\Omega l}{\cos\psi}\,,$$

$$\dot\varphi = \frac{v + \Omega l\sin\psi}{R\cos\psi}\,.$$



with delay???

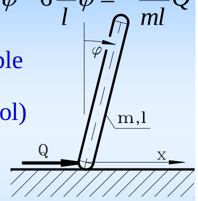
3. Balancing

$$\ddot\varphi - 6\frac{g}{l}\varphi = -\frac{6}{ml}Q$$

1) $Q = 0$ - no control

$$\ddot\varphi - 6\frac{g}{l}\varphi = 0 \;\Rightarrow\; \varphi = 0 \text{ is unstable}$$

2) $Q(t) = P\varphi(t) + D\dot\varphi(t)$ (PD control)



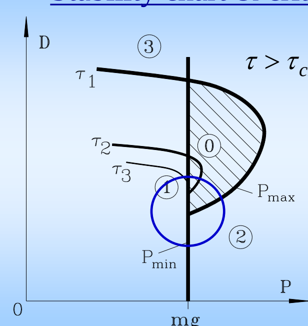
$$\ddot\varphi + \frac{6}{ml}D\dot\varphi + \frac{6}{ml}(P - mg)\varphi = 0$$

$\varphi = 0$ is asympt. stable $\Leftrightarrow D > 0,\; P > mg$

3) $Q(t) = P\varphi(t-\tau) + D\dot\varphi(t-\tau)$ (with reflex delay τ)

$$\ddot\varphi(t) + \frac{6}{ml}D\dot\varphi(t-\tau) + \frac{6}{ml}P\varphi(t-\tau) - \frac{6g}{l}\varphi(t) = 0$$

Stability chart & critical reflex delay



$\tau > \tau_{cr} = \sqrt{\frac{l}{3g}} \Rightarrow$ instability

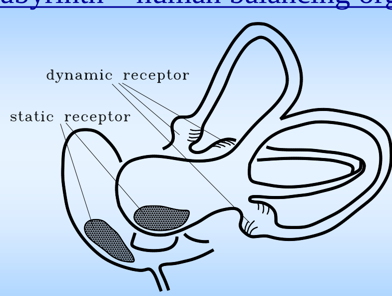
$l = 0.3\,[\text{m}] \Rightarrow$

$\tau \approx 0.1\,[\text{s}]$

$0 < \frac{\omega}{2\pi\tau} = f <$

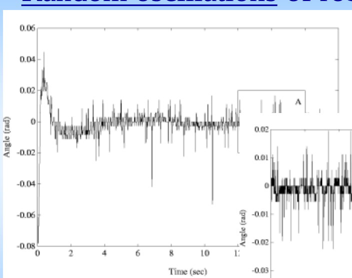
$< \frac{1}{4\tau} \approx 2.5\,[\text{Hz}]$

Labyrinth – human balancing organ

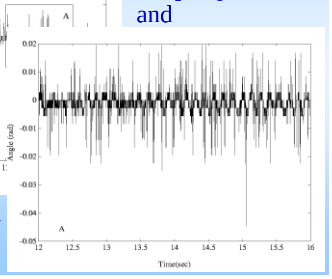


Both angle and angular velocity signals are needed

Random oscillations of robotic balancing

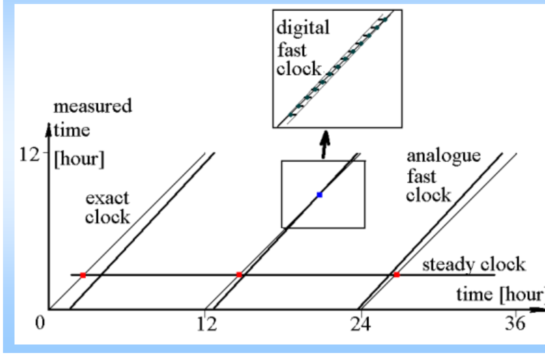


sampling time and

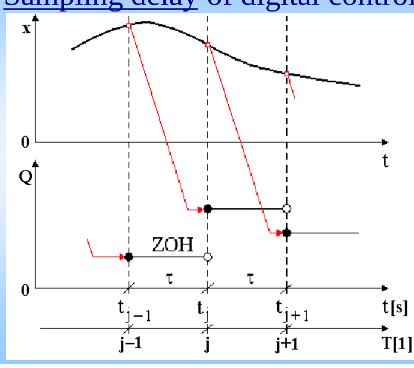


quantization (round-off)

Alice's Adventures in Wonderland



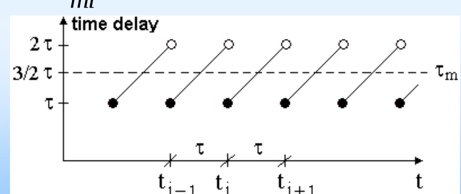
Sampling delay of digital control



Digitally controlled pendulum

$$\ddot{\varphi}(t) - \frac{6g}{l}\varphi(t) = u_j, \quad t \in [t_j, t_j + \tau)$$

$$u_j = -\frac{6}{ml}\left(D\dot{\varphi}(t_j - \tau) + P\varphi(t_j - \tau)\right) \quad j = 1,2,\ldots$$

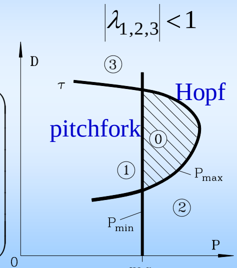


Stability of digital control – sampling

$$\mathbf{x}^j = \begin{pmatrix} \varphi(t_j) \\ \varphi(t_j) \\ u_j \end{pmatrix} \qquad \mathbf{x}^{j+1} = \mathbf{A}\mathbf{x}^j \qquad \det(\lambda\mathbf{I} - \mathbf{A}) = 0$$

$$\omega = \tau\sqrt{\frac{6g}{l}} \qquad \left|\lambda_{1,2,3}\right| < 1$$

$$\mathbf{A} = \begin{pmatrix} \operatorname{ch}\omega & \frac{\operatorname{sh}\omega}{\omega} & \frac{\operatorname{ch}\omega - 1}{\omega^2} \\ \omega\operatorname{sh}\omega & \operatorname{ch}\omega & \frac{\operatorname{sh}\omega}{\omega} \\ -\frac{6\tau^2}{ml}P & -\frac{6\tau}{ml}D & 0 \end{pmatrix}$$



Stability of digital control – round-off

h – one digit converted to control force

$$u_j = -\frac{6}{ml} h \operatorname{int}\left(\frac{D\varphi(t_j - \tau) + P\varphi(t_j - \tau)}{h}\right)$$

$$\mathbf{x}^{j+1} = \mathbf{B}\mathbf{x}^j + \mathbf{g}(\mathbf{x}^j)$$

$$\mathbf{g}(\mathbf{x}^j) = \begin{pmatrix} 0 \\ 0 \\ -\frac{6\tau^2}{ml} h \operatorname{int}\left(\frac{P}{h}x_1^j + \frac{D}{\tau h}x_2^j\right) \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} \operatorname{ch}\omega & \frac{\operatorname{sh}\omega}{\omega} & \frac{\operatorname{ch}\omega - 1}{\omega^2} \\ \omega\operatorname{sh}\omega & \operatorname{ch}\omega & \frac{\operatorname{sh}\omega}{\omega} \\ 0 & 0 & 0 \end{pmatrix}$$

$\det(\lambda\mathbf{I} - \mathbf{B}) = 0 \Rightarrow$

$\lambda_1 = e^{\omega} > 1$, $\lambda_2 = e^{-\omega}$, $\lambda_3 = 0$

1D cartoon – the μ-chaos map

Drop 2 dimensions, rescale x with h $\Rightarrow$ $a \sim e^{\omega}$, $b \sim P$

$$x_{j+1} = ax_j - b\,\mathrm{int}(x_j)$$

A pure mathematical approach ($p > 0$, $p < q$)

$$\dot{y}(t) = py(t) - q\,\mathrm{int}\big(y(\mathrm{int}(t))\big)$$

solution with $x_j = y(j)$ leads to μ-chaos map,

$a = e^p$, $b = q(e^p - 1)/p$ $\Rightarrow$ $a > 1$, $(0 <)\ a - b < 1$

small scale: $x_{j+1} = a\, x_j$, large scale: $x_{j+1} = (a - b)\, x_j$

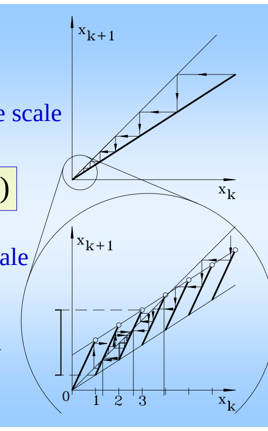
Micro-chaos map

large scale

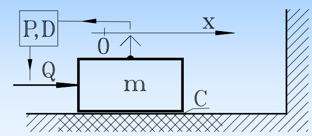
$$x_{k+1} = ax_k - b\,\mathrm{int}(x_k)$$

small scale

Typical in digitally controlled machines, caused partly by delay



4. Robotic position control



Equation of motion

$$m\ddot{x} = Q - C\,\mathrm{sgn}\,\dot{x}, \quad Q = -D\dot{x} - Px$$

Position error: $\Delta = C / P$, Stability $\Leftrightarrow P > 0,\ D > 0$

With sampling delay τ, dimensionless time $T = t / \tau$

$$x''(T) = -\frac{P\tau^2}{m}x(j-1) - \frac{D\tau}{m}x'(j-1), \quad T \in [j, j+1)$$

Stability of digital position control

$$x''(T) \equiv \underbrace{-px(j-1) - dx'(j-1)}_{=:\,a_j}, \quad T \in [j, j+1)$$

$$x(T) = x(j) + x'(j)(T-j) + \tfrac{1}{2}a_j(T-j)^2$$

$$x'(T) = x'(j) + a_j(T-j), \quad T \in [j, j+1)$$

$$\mathbf{z}^j := \begin{pmatrix} x(j) \\ x'(j) \\ a_j \end{pmatrix} \Rightarrow \mathbf{z}^{j+1} = \mathbf{A}\mathbf{z}^j, \quad \mathbf{A} = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ -p & -d & 0 \end{pmatrix}$$

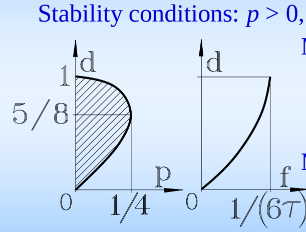
$$\det(\mu\mathbf{I} - \mathbf{A}) = \mu^3 - 2\mu^2 + (1 + d + \tfrac{1}{2}p)\mu + (\tfrac{1}{2}p - d) = 0$$

Stability chart

$$\mathrm{Re}\,\eta_{1,2,3} < 0$$

$$p\eta^3 + 2(d-p)\eta^2 + (4 - 4d + p)\eta + 2(2+d) = 0$$

Stability conditions: $p > 0$, $H_2 > 0$ ($= 0 \Rightarrow$ Hopf)



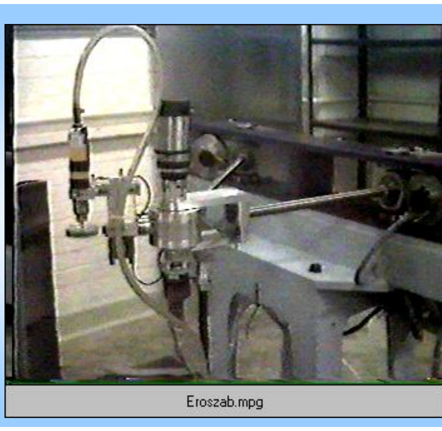
Maximum gain:

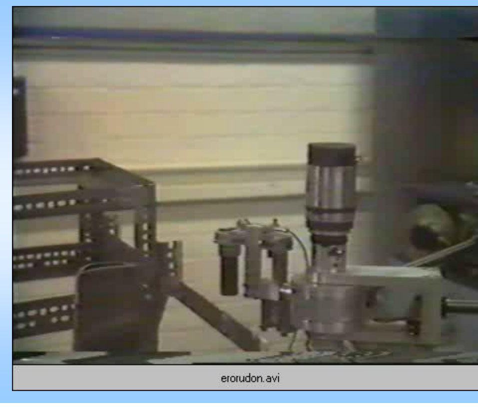
$$P_{\max} = \frac{1}{4}\frac{m}{\tau^2}$$

Minimum position error

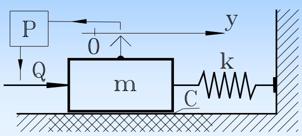
$$\Delta_{\min} \geq 4C\tau^2 / m$$

Self-excited vibration frequency: $0 < f < f_{\mathrm{sampling}}/6$





5. Robotic force control



$$m\ddot{y} + ky = -P(ky - F_d) + ky - C\,\mathrm{sgn}\,\dot{y}$$

Equilibrium: $y_d = F_d / k$, Force error: $\Delta_F = C/P$

Stability $\Leftrightarrow$ $P > 0.$ But, with sampling delay τ

$$Q(t) \equiv -P\big(ky(t_j - \tau) - F_d\big) + ky(t_j - \tau),\ t \in [t_j, t_j + \tau)$$

Dimensionless parameters: $(\omega_n \tau)/(2\pi) = f_n / f_s,\ P$

Stability chart of force control

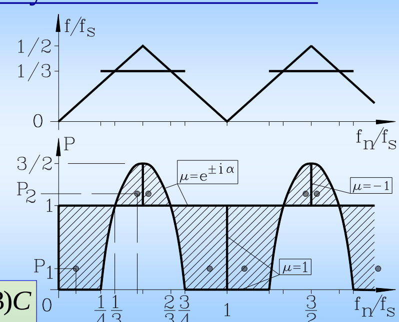
Vibration frequency: $0 < f < f_s/2$

Maximum gain:

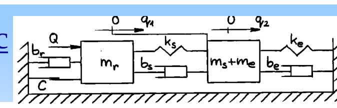
$P_{max} = 1.5$

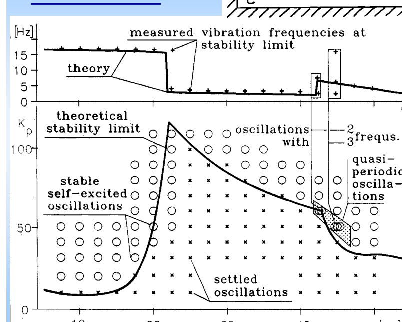
Minimum force error:

$\Delta_{F,\min} \geq (2/3)C$



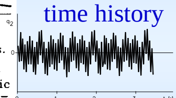
Quasi-periodic oscillation

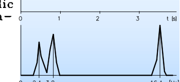




2 DoF model

time history

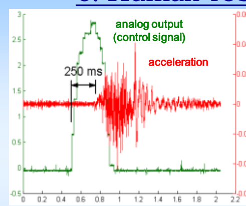




spectrum

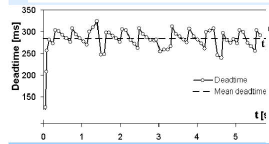
stability chart

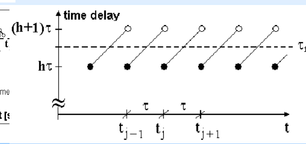
6. Human-robotic force control



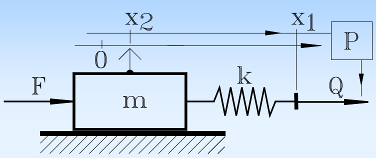
Random delay measurements

Combined pure delay and sampling





Modelling human-robotic force control



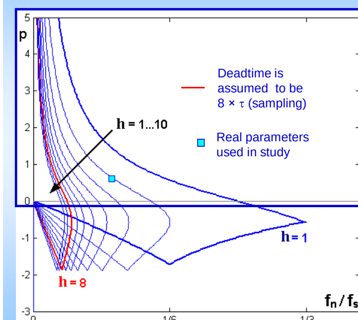
$$m\ddot{x}_2(t) = k(x_1(t) - x_2(t)), \qquad t \in [t_j, t_j + \tau)$$

$$\dot{x}_1(t) = -Pk\underbrace{\left(x_1(t_j - h\tau) - x_2(t_j - h\tau)\right)} + \dot{x}_2(t_j - h\tau)$$

force error: $F_e(t_j - h\tau)$

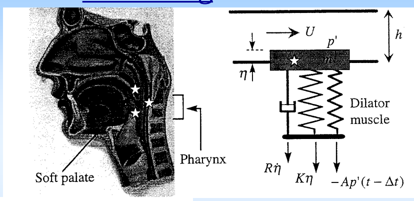
dimensionless gain: $p = Pk/\omega_n$

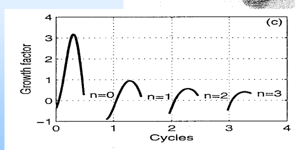
Stability of rehabilitation robot



The unmodelled damping of the human palm and other viscous effects help to stabilize at chosen parameters

7. Snoring





Ffowcs
(Cambridge, 1997)

8. Human-human force control

Conclusion

How does delay arise in Engineering?

By elastic-plastic contact (PDE⇒RFDE)
By information lag in control

Thank you for your attention!